QUANTITATIVE ILLUMINATION OF CONVEX BODIES AND VERTEX DEGREES OF GEOMETRIC STEINER MINIMAL TREES

KONRAD J. SWANEPOEL

ABSTRACT. In this note we prove two results on the quantitative illumination parameter $f(d)$ of the unit ball of a d-dimensional normed space introduced by K. Bezdek (1992). The first is that $f(d) = O(2^d d^2 \log d)$. The second involves Steiner minimal trees. Let $v(d)$ be the maximum degree of a vertex, and $s(d)$ of a Steiner point, in a Steiner minimal tree in a d-dimensional normed space, where both maxima are over all norms. F. Morgan (1992) conjectured that $s(d) \leq 2^d$, and D. Cieslik (1990) conjectured $v(d) \leq 2(2^d - 1)$. We prove that $s(d) \leq v(d) \leq f(d)$ which, combined with the above estimate of $f(d)$, improves the previously best known upper bound $v(d) < 3^d$.

1. INTRODUCTION

Let K denote a convex body in the d-dimensional real vector space $\mathbf{R}^d$. Denote its volume by $\mu(K)$ and its translative covering density by $\vartheta(K)$. A (positive) *homothet* with ratio $\lambda > 0$ of K is any set of the form $\lambda K + t$, with $t \in \mathbf{R}^d$. The *difference body* of K is $K - K$. According to the Rogers-Shephard inequality [RS57], $\mu(K-K)/\mu(K) \leq \binom{2d}{d}$. If K is *centred* (that is, $K = -K$), then of course $\mu(K-K)/\mu(K) = 2^d$, and K defines a norm

$$\|x\|_K := \inf\{\lambda > 0 : \lambda^{-1}x \in K\},$$

which turns $\mathbf{R}^d$ into a normed space. Let $\mathcal{K}^d$ denote the class of all d-dimensional convex bodies, and $\mathcal{K}^d_o$ the class of all centred d-dimensional convex bodies.

1.1. Quantitative illumination and covering.

A point $p \notin K$ *illuminates* a point q on the boundary of K if the ray

$$\{\lambda p + (1-\lambda)q : \lambda < 0\}$$

intersects the interior of K. A set of points $P \subseteq \mathbf{R}^d \setminus K$ *illuminates* K if each boundary point of K is illuminated by some point in P. Let $L(K)$ be the smallest size of a set that illuminates K. Also let $L(d) := \max\{L(K) : K \in \mathcal{K}^d\}$, and $L_o(d) := \max\{L(K) : K \in \mathcal{K}^d_o\}$. Since $L(K) = 2^d$ if K is a cube, $L(d) \geq L_o(d) \geq 2^d$. The well-known illumination problem is to show that $L(d) = 2^d$. For large d the best known upper bounds are $L(d) \leq \binom{2d}{d} d(\log d + \log\log d + 5)$ and $L_o(d) \leq 2^d d(\log d + \log\log d + 5)$, due to Rogers [Grü63, p. 284]; see also [RZ97].

There are other equivalent formulations of this illumination problem. For example, let $L'(K)$ be the smallest number of positive homothets of

K, with each homothety ratio less than 1, whose union contains K. Then $L(K) = L'(K)$. See [MS99] for a survey on this problem and its history.

We consider quantitative versions of the above two formulations of the illumination problem. The first was introduced by K. Bezdek [Bez92]. For $K \in \mathcal{K}_o^d$ let

$$B(K) := \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K \right\}.$$

This ensures that far-away light sources are penalised. Let

$$B(d) := \sup\{B(K) : K \in \mathcal{K}_o^d\}.$$

Bezdek asked for the value of $B(d)$, and in particular, if $B(d)$ is finite for $d \geq 3$. He showed that $B(2) = 6$; the regular hexagon giving equality. Note that $B(K) \geq L(K)$, hence $B(d) \geq L_o(d) \geq 2^d$. It is also easily seen that $B(K) = 2^d$ if K is a d-cube, and $B(K) = 2d$ if K is a d-cross polytope.

We introduce the following quantitative covering parameter for $K \in \mathcal{K}^d$:

$$C(K) := \inf \left\{ \sum_i (1-\lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbf{R}^d \right\}.$$

In this way homothets almost as large as K are penalised.

Proposition 1. *For any $K \in \mathcal{K}_o^d$ we have $B(K) \leq 2C(K)$.*

Let

$$C(d) := \sup\{C(K) : K \in \mathcal{K}^d\},$$

and

$$C_o(d) := \sup\{C(K) : K \in \mathcal{K}_o^d\}.$$

Hence $C(d) \geq C_o(d) \geq B(d)/2$. It is easy to see that $C(K) = 2^{d+1}$ if K is a d-cube, hence $C(d) \geq C_o(d) \geq 2^{d+1}$. As before, it is not clear whether $C(d)$ is finite. Levi [Lev54] showed that any planar convex body can be covered with 7 homothets, each with homothety ratio $1/2$; hence $C(2) \leq 14$. Lassak's result [Las86] that any planar convex body can be covered with 4 homothets, each with ratio $1/\sqrt{2}$, improves this to $C(2) \leq 8 + 4\sqrt{2}$. Lassak [Las98] also showed that any convex body in $\mathbf{R}^3$ can be covered with 28 homothets, each with ratio $7/8$; hence $C(3) \leq 224$. We show that a result of Rogers and Zong [RZ97] implies the following upper bound.

Theorem 1. *For any d-dimensional convex body K we have*

$$C(K) < e(d+1)\frac{\mu(K-K)}{\mu(K)}\vartheta(K).$$

Using Rogers' estimate [Rog57] $\vartheta(K) \leq d(\log d + \log\log d + 5)$ for $d \geq 2$ and the Rogers-Shephard inequality one finds

$$C(d) < \binom{2d}{d} e(d+1)d(\log d + \log\log d + 5) = O(4^d d^{3/2} \log d),$$

and

$$B(d) \leq 2C_o(d) < 2^{d+1} e(d+1)d(\log d + \log\log d + 5) = O(2^d d^2 \log d).$$

Perhaps $C(d) = O(2^d)$.

1.2. **Steiner minimal trees.** Given a finite set of points V in $\mathbf{R}^d$, a *Steiner tree* T of V is any tree in $\mathbf{R}^d$ whose vertex set contains V, and whose edges are straight-line segments in $\mathbf{R}^d$. The vertices of T not in V are called *Steiner points*. (Usually Steiner points are required to have degree at least 3, but this is unnecessary here.) The *K-length* of a Steiner tree is the total length in $\|\cdot\|_K$ of the edges of the tree, where K is a centred convex body. It is easily seen [Coc67] that any given point set has a Steiner tree of smallest K-length, called a *K-Steiner minimal tree* (K-SMT).

Steiner minimal trees have been studied mostly in the Euclidean plane and the rectilinear plane (K a parallelogram) [HRW92]. Other normed planes have also been considered; see [Bra01, §3.1] for further references. Steiner minimal trees in normed spaces of higher dimension have been investigated by Cieslik [Cie98] and Morgan [Mor92] among others.

Let $v(K)$ be the maximum possible degree of a vertex in a K-SMT, and $s(K)$ the maximum possible degree of a Steiner point in a K-SMT. Clearly $s(K) \leq v(K)$. The following table gives some examples of known values of $s(K)$ and $v(K)$. See [Swa99, Swa00, BTW00] for further examples.

K	$s(K)$	$v(K)$
Euclidean d-ball	3	3
d-cube	2^d	2^d
d-cross polytope	$2d$	$2d$
regular hexagon	4	6

Let $s(d) := \max\{s(K) : K \in \mathcal{K}_o^d\}$, and $v(d) := \max\{v(K) : K \in \mathcal{K}_o^d\}$. Then $2^d \leq s(d) \leq v(d)$. The following two conjectures have been made:

Conjecture 1 (Cieslik [Cie90], [Cie98, ch. 4])**.** *$v(d) \leq 2(2^d - 1)$ for all $d \geq 2$.*

Conjecture 2 (Morgan [Mor92], [Mor98, ch. 10])**.** *$s(d) \leq 2^d$ for all $d \geq 2$.*

Cieslik [Cie90] has shown that $v(K) \leq H(K)$ where $H(K)$ is the translative kissing number of K. See [Zon98] for a survey and for references to the following upper bounds on $H(K)$. Since $H(K) \leq 3^d - 1$ with equality only for (affine images of) the d-cube, it follows that $v(d) \leq 3^d - 2$. Since for planar K we have $H(K) \leq 6$ if K is not a parallelogram, we obtain $v(2) = 6$ [Cie90]; thus Conjecture 1 is true for $d = 2$. Conjecture 2 is also true for $d = 2$ [Swa00]. The two-dimensional methods are very special and offer no hope for generalisation to higher dimensions. We find upper bounds within a factor of $O(d^2 \log d)$ from the conjectured values, using the following relationship with Bezdek's illumination parameter.

Theorem 2. *For any $K \in \mathcal{K}_o^d$ we have $v(K) \leq B(K)$.*

Note that equality holds, for example, if K is a regular hexagon, a d-cube or a d-cross polytope, but not if K is a d-ball.

Corollary 1. *For any $K \in \mathcal{K}_o^d$ we have $s(K) \leq v(K) < 2^{d+1} e(d+1)\vartheta(K)$.*

Corollary 2. $s(d) \leq v(d) = O(2^d d^2 \log d)$.

2. PROOFS

Proof of Proposition 1. Let $\{\lambda_i K + t_i\}$ be a finite covering of K, with $0 < \lambda_i < 1$ for all i. Let $\varepsilon > 0$ be sufficiently small such that all $\lambda_i + \varepsilon < 1$. If a boundary point q of K is covered by $\lambda_i K + t_i$, then $1 - \lambda_i \leq \|t_i\|_K \leq 1 + \lambda_i < 2$, and the centre of the homothety mapping K to $(\lambda_i + \varepsilon)K + t_i$, namely $p_i := (1 - \lambda_i - \varepsilon)^{-1} t_i$, is outside K and illuminates q. Therefore, the set $\{p_i\}$ illuminates K, and $\sum_i \|p_i\|_K < \sum_i 2/(1 - \lambda_i - \varepsilon)$. Since $\varepsilon > 0$ can be made arbitrarily small, $\sum_i \|p_i\|_K \leq 2 \sum_i (1 - \lambda_i)^{-1}$. □

Proof of Theorem 1. It is known [RZ97] that for any $0 < \lambda < 1$ there exists a covering of K by homothets $\{\lambda K + t_i : i = 1, \dots, N\}$, with

$$N \leq \frac{\mu(K - \lambda K)}{\mu(\lambda K)} \vartheta(K) < \lambda^{-d} \frac{\mu(K - K)}{\mu(K)} \vartheta(K).$$

Choosing $\lambda = d/(d+1)$ we obtain

$$\sum_{i=1}^{N} (1 - \lambda)^{-1} < (d+1) \left(1 + \frac{1}{d}\right)^d \frac{\mu(K - K)}{\mu(K)} \vartheta(K) < (d+1)e \frac{\mu(K - K)}{\mu(K)} \vartheta(K).$$

□

Lemma 1. *If p illuminates the boundary point u of $K \in \mathcal{K}_o^d$, then for all sufficiently small $\varepsilon > 0$, $\|u - \varepsilon p\|_K < 1 - \varepsilon$.*

Proof. The lemma is trivial if $p = \lambda u$ for some λ. Therefore, assume that p and u are linearly independent and consider the two-dimensional subspace spanned by them (Figure 1). Since p illuminates u, we may choose $\varepsilon_0 > 0$

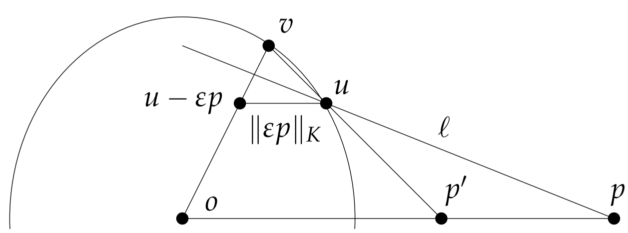

FIGURE 1.

such that the line through o and $u - \varepsilon_0 p$ intersects the line ℓ through u and p in the interior of K. Then clearly for all $\varepsilon > 0$ with $\varepsilon < \varepsilon_0$ the line through o and $u - \varepsilon p$ still intersects ℓ in the interior of K. Let $v = (\|u - \varepsilon p\|_K)^{-1}(u - \varepsilon p)$. Then the lines vu and op intersect in p', say, with $\|p'\|_K < \|p\|_K$. Using similar triangles, $\|u - \varepsilon p\|_K = 1 - \|\varepsilon p\|_K / \|p'\|_K < 1 - \varepsilon$. □

Proof of Theorem 2. Consider a vertex of a K-SMT of degree $v(K)$. By translating we may assume that the vertex is the origin o. By scaling we may also assume that each edge emanating from o has K-length at least 1. Let these edges be ov_i, with $\|v_i\|_K \geq 1$. Let $u_i = \|v_i\|_K^{-1} v_i$. Then the star T joining o to each u_i is a K-SMT of $\{o, u_1, u_2, \dots, u_{v(K)}\}$ (otherwise we would be able to shorten the original tree).

Let $\{p_1, \dots, p_k\}$ illuminate K. For each $j = 1, \dots, k$, let

$$U_j = \{u_i : p_j \text{ illuminates } u_i\}.$$

Then $\{u_i\} = \bigcup_j U_j$. We estimate the number of points $|U_j|$ in each U_j. By Lemma 1 we may find $\varepsilon > 0$ such that $\|u_i - \varepsilon p_j\|_K < 1 - \varepsilon$ for all i. Consider the tree T' obtained from the star T by replacing, for each $u_i \in U_j$, the edge from o to u_i by the edge from εp_j to u_i, and joining the Steiner point εp_j to o. Then T' is not shorter than T. This implies that

$$\begin{aligned} |U_j| = \sum_{u_i \in U_j} \|u_i\|_K &\leq \|\varepsilon p_j\|_K + \sum_{u_i \in U_j} \|u_i - \varepsilon p_j\|_K \\ &< \varepsilon \|p_j\|_K + (1-\varepsilon)|U_j|, \end{aligned}$$

and $|U_j| < \|p_j\|_K$. Hence $v(K) \leq \sum_{j=1}^k |U_j| < \sum_{j=1}^k \|p_j\|_K$. Taking the infimum over all sets $\{p_i\}$ that illuminate K, we obtain that $v(K) \leq B(K)$. $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF SOUTH AFRICA, PO BOX 392, PRETORIA 0003, SOUTH AFRICA

E-mail address: swanekj@unisa.ac.za